Alternating Series :

A series whose terms are alternately +ve and -ve, is called an alternating series, i.e. a series of the form

$$u_1 - u_2 + u_3 - u_4 + \cdots \cdots, \quad u_n > 0 \ \forall n$$

is an alternating series.

The following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

$$1 - 1 + 1 - 1 + \cdots$$

are alternating series.

Leibnitz's theorem: In an alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$, if (i) $u_1 > u_2 > u_3 > \cdots$ (ii) $\lim_{n\to\infty} u_n = 0$

then series is convergent, its sum is +ve and does not exceed the first terms.

Proof: The series $u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots$

Can be written in each of the following ways:

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots \quad \text{—— (I)}$$

and $$u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots \quad \text{—— (II)}$$

From (I),

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}) \quad \text{—— (III)}$$

and from (II)

$$S_{2n+1} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n} - u_{2n+1}) \quad \text{—— (iv)}$$

From (III) and (iv); we have

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$$S_{2n} = S_{2n+1} - u_{2n+1} \quad \text{———} \quad (v)$$

Since in (iii), $(u_1 - u_2), (u_3 - u_4), \ldots$ are +ve, hence S_{2n} is +ve.

Also in (iv) each term in brackets in positive.

Hence the value of S_{2n+1} decreases with the increas of n and always less than u_1.

From (v),

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

But as $n \to \infty$, $u_{2n+1} \longrightarrow 0$

Therefore, $\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_n$

Hence the sum of the series is always > 0 and $< u_1$ and therefore converges to a definite value between 0 & u_1 i.e. the series is convergent.

Example:- The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent.

The series is an alternating series and each term is numerically less than its proceeding term and $\lim_{n \to \infty} u_n = \pm \lim_{n \to \infty} \frac{1}{n} = 0$, Hence the series is convergent.

Absolute Convergent Series:- Let the series be $\Sigma u_n = u_1 + u_2 + \ldots$ in which any term may be either +ve or −ve. Let $|u_n|$ denote the absolute value of u_n i.e. $|u_n| = u_n$ if u_n is +ve and $|u_n| = -u_n$ if u_n is −ve.

Then

$$\sum|u_n| = |u_1|+|u_2|+|u_3|+\cdots+|u_n|+\cdots$$

The series $\sum u_n$ containing positive and negative terms is said to be absolutely convergent if the series $\sum|u_n|$ is convergent.

Examples – The series $1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$ is absolutely convergent.

Here $\sum u_n = 1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$

Then $\sum|u_n| = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ which is convergent.

Hence $\sum u_n$ is absolutely convergent.

Semi convergent or Conditionally convergent Series –

If the series $\sum u_n$ is convergent but the series $\sum|u_n|$ is divergent, then the series $\sum u_n$ is said to be semi-convergent.

Example – The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is semi-convergent.

$\sum u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent, but $\sum|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ divergent. Hence $\sum u_n$ is semi-convergent.

Theorem – A series which is absolutely convergent, it itself convergent but the converse is not true.

Proof – Let the series $\sum u_n$ be absolutely convergent then by definition of absolute convergent, $\sum|u_n|$ is convergent.

Now $u_n+|u_n| = 2u_n$ or 0 according u_n is +ve or −ve.

Therefore, every term of the series $\sum(u_n+|u_n|)\geq 0$ and $\leq$ the corresponding term of the series $\sum 2|u_n|$, which is convergent.

Hence $\sum(u_n+|u_n|)$ is convergent and consequently $\sum u_n$ is convergent; but the converse is not true; for example – The series $\sum u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent but $\sum|u_n|$ is divergent.

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Binomial Series :-

The series $1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{\lfloor r}\cdot x^r+\cdots$

where n is any given number, is a binomial series.

When n is +ve integer, the given series terminate after $(n+1)$th terms.

In all other cases the series is infinite and we will discuss the convergence in this case only that is when n is not +ve integer.

To show the series

$$1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{\lfloor r}x^r+\cdots$$

is convergent for all numerical value of x less than 1 when n is not +ve integer.

Let us denote the series by $u_0+u_1+\cdots+u_r+u_{r+1}+\cdots$

Therefore.
$$u_{r+1}=\frac{n(n-1)\cdots(n-r)}{\lfloor r+1}\cdot x^{r+1}$$

& $$u_r=\frac{n(n-1)\cdots(n-r+1)}{\lfloor r}x^r$$

Therefore.
$$\frac{u_r}{u_{r+1}}=\frac{n(n-1)\cdots(n-r+1)x^r}{\lfloor r}\times\frac{\lfloor r+1}{n(n-1)\cdots(n-r+1)(n-r)\cdot x^{r+1}}$$

$$\frac{u_r}{u_{r+1}}=\left(\frac{r+1}{n-r}\right)\cdot\frac{1}{x}$$

Therefore, $\lim_{r\to\infty} \frac{u_r}{u_{r+1}} = \lim_{r\to\infty} \frac{r(1+\frac{1}{r})}{-r(1-\frac{n}{r})} \times \frac{1}{x} = -\frac{1}{x}$

Now $\lim_{r\to\infty} \left| \frac{u_r}{u_{r+1}} \right| = \left| -\frac{1}{x} \right| = \frac{1}{x}$

Hence the series is absolutely convergent when $\frac{1}{|x|} > 1$ or $|x| < 1$ i.e. the series is convergent if $|x| < 1$ i.e. the series is convergent when the numerical value of x is less than 1.

Exponential Series :-

To show that the series $1 + x + \frac{x^2}{|\underline{2}} + \cdots + \frac{x^n}{|\underline{n}} + \cdots$ is convergent for all values of x.

Let us denote the series by $u_1 + u_2 + \cdots + u_n + \cdots$

Therefore. $u_n = \frac{x^n}{|\underline{n}}$ and $u_{n+1} = \frac{x^{n+1}}{|\underline{n+1}}$

Now $\frac{u_n}{u_{n+1}} = (n+1) \cdot \frac{1}{x}$

(i) When $x > 0$, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty > 1$, Hence the series is convergent.

(ii) When $x < 0$, $\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim_{n\to\infty} (n+1) \cdot \frac{1}{|x|} = \infty > 1$

Hence the series is convergent.

(iii) When $x = 0$, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty > 1$

and given series is convergent.

Hence the exponential series is convergent $\forall x$.

Logarithmic Series:-

To discuss the convergency of the series
$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\cdot\frac{x^n}{n} + \cdots$

Let us denote the series by $u_1 + u_2 + \cdots + u_n + \cdots$

Therefore.
$$u_n = (-1)^{n+1}\cdot\frac{x^n}{n}$$

and
$$u_{n+1} = (-1)^{n+2}\cdot\frac{x^{n+1}}{n+1}$$

Hence
$$\frac{u_n}{u_{n+1}} = (-1)^{n+1}\cdot\frac{x^n}{n} \times \frac{n+1}{(-1)^{n+2}\cdot x^{n+1}}$$
$$= \left(1+\frac{1}{n}\right)\left(-\frac{1}{x}\right)$$

Therefore, $\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(\frac{1}{|x|}\right) = \frac{1}{|x|}$

Hence the series is absolutely convergent if $\frac{1}{|x|} > 1$.

If $x=1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
which is convergent as we have discussed in the case of alternating series.

If $x=-1$, the series becomes
$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$
which is divergent as we have discussed.